The dispersive self-dual Einstein equations and the Toda lattice

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Abstract. The Boyer–Finley equation, or $SU(\infty)$ Toda equation is both a reduction of the self-dual Einstein equations and the dispersionless limit of the 2D Toda lattice equation. This suggests that there should be a dispersive version of the self-dual Einstein equation which both contains the Toda lattice equation and whose dispersionless limit is the familiar self-dual Einstein equation. Such a system is studied in this paper. The results are achieved by using a deformation, based on an associative $\star$-product, of the algebra sdiff(Σ^2) used in the study of the undeformed, or dispersionless, equations.

1. Introduction

One much studied $(2 + 1)$-dimensional integrable system, known as the Boyer–Finley equation [1, 2], is

$$\nabla^2 \rho = \partial_x^2 e^\rho \tag{1}$$

where $\rho = \rho(z, \tilde{z}, x)$ and $\nabla^2 = \partial_z \partial_{\tilde{z}}$. This also has been referred to, for reasons that will be explained later, as the $SU(\infty)$ Toda or sdiff(Σ^2) Toda equation. This equation has two interesting properties. Firstly it is a reduction of the self-dual Einstein equations (the equations governing any metric with self-dual Weyl tensor and vanishing Ricci tensor) under a rotational Killing vector. Secondly it is the dispersionless (or long-wave or continuum) limit of the Toda lattice equation

$$\nabla^2 \rho_n = e^{\rho_{n+1}} - 2e^{\rho_n} + e^{\rho_{n-1}} \tag{2}$$

where $\rho_n = \rho_n(z, \tilde{z})$. Heuristically, solutions whose natural length scale is large compared with the lattice spacing do not see the lattice, and the difference operator on the right-hand side may be approximated by a second derivative. These two properties may be summarized in the following diagram:

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$$\begin{pmatrix} \text{Toda lattice} \\ \text{equation} \\ \text{(three dimensions)} \end{pmatrix}$$

$\downarrow$ dispersionless limit

$$\begin{pmatrix} \text{self-dual Einstein} \\ \text{equation} \\ \text{(four dimensions)} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{Boyer–Finley} \\ \text{equation} \\ \text{(four dimensions)} \end{pmatrix}$$

reduction

This diagram suggests that there should be some system whose dispersionless limit is the self-dual Einstein equation, and which contains the Toda lattice equation embedded within it as a special case. This system, the subject of this paper, will be called the dispersive self-dual Einstein equation (such a system may not be unique, see section 5). This will have exactly those properties needed to make the diagram in figure 1 commute.
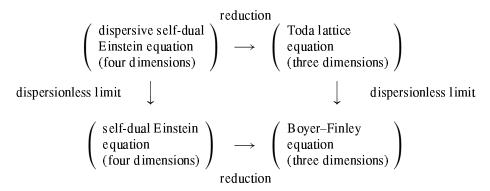
reduction

$$\begin{pmatrix} \text{dispersive self-dual} \\ \text{Einstein equation} \\ \text{(four dimensions)} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{Toda lattice} \\ \text{equation} \\ \text{(three dimensions)} \end{pmatrix}$$

dispersionless limit $\downarrow$ $\qquad\qquad\qquad$ $\downarrow$ dispersionless limit

$$\begin{pmatrix} \text{self-dual Einstein} \\ \text{equation} \\ \text{(four dimensions)} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{Boyer–Finley} \\ \text{equation} \\ \text{(three dimensions)} \end{pmatrix}$$

reduction

Figure 1.

Such a system was introduced by the author [3] and studied further in [4–6]. These papers, however, do not study the relationship between the dispersive self-dual Einstein equation and the Toda lattice equation.

2. The self-dual Einstein equations

There are several ways of defining the equations governing a self-dual vacuum metric. One particular useful method is the following [7]. Let $\mathcal{U} = V_1 + \lambda V_2$ and $\mathcal{V} = V_3 + \lambda V_4$ be two commuting vectors fields (for all values of the spectral parameter λ, and where the V_i are independent of λ) on some four-manifold $\mathcal{M}$. Suppose further that each V_i preserves a four-form ω on $\mathcal{M}$. Then the contravariant metric

$$\mathbf{g} = \Lambda^{-1}[V_1 \otimes_S V_4 - V_2 \otimes_S V_3] \tag{3}$$

where the conformal factor is defined by

$$\Lambda = \omega(V_1, V_2, V_3, V_4)$$

is a self-dual metric. It may also be shown that all self-dual metrics may be written in this way [7]. One approach that has attracted much attention is to write $\mathcal{M} = R^2 \times \Sigma^2$

and to take the vectors fields to belong to sdiff (Σ^2), the Lie algebra of volume preserving diffeomorphisms of the 2-surface Σ^2. Explicitly, write

$$X_f = \frac{\partial f(z, \tilde{z}, x, p)}{\partial x} \frac{\partial}{\partial p} - \frac{\partial f(z, \tilde{z}, x, p)}{\partial p} \frac{\partial}{\partial x}.$$

Here z and $\tilde{z}$ are coordinates on R^2 and x and p are coordinates on Σ^2. These vector fields obey the important relation $[X_f, X_g] = X_{\{f,g\}}$, where $\{f, g\}$ is the Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial x}.$$

With the vector fields

$$\mathcal{U} = \partial_z + \lambda X_f$$

$$\mathcal{V} = \partial_{\tilde{z}} + \lambda X_g$$

and 4-form $\omega = \mathrm{d}z \wedge \mathrm{d}\tilde{z} \wedge \mathrm{d}x \wedge \mathrm{d}p$ one obtains Plebanski's form of the self-duality equations [8]:

$$\Omega_{zx}\Omega_{\tilde{z}p} - \Omega_{zp}\Omega_{\tilde{z}x} = 1$$

or, using the Poisson bracket,

$$\{\Omega_z, \Omega_{\tilde{z}}\} = 1. \tag{4}$$

Other choices lead to other, equivalent, forms of the self-duality equations. For example, with the vector fields (where the functions f, g and $h_\pm$ depend on the four variables z, $\tilde{z}$, x and p)

$$V_1 = \partial_z + X_f$$

$$V_2 = X_{h_+}$$

$$V_3 = X_{h_-}$$

$$V_4 = -\partial_{\tilde{z}} - X_g$$

which preserve the 4-form

$$\omega = \mathrm{d}\tilde{z} \wedge \mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}p$$

one obtains, from the condition the vector fields $\mathcal{U} = V_1 + \lambda V_2$ and $\mathcal{V} = V_3 + \lambda V_4$ commute for all values of λ, the equations

$$\partial_z h_- + \{f, h_-\} = 0 \tag{5}$$

$$\partial_{\tilde{z}} h_+ + \{g, h_+\} = 0 \tag{6}$$

$$\partial_z g - \partial_{\tilde{z}} f + \{f, g\} - \{h_+, h_-\} = 0. \tag{7}$$

Hence, using the theorem proved in [7] (see also [9]), any solution of this system generates a self-dual metric, one particular form of which is given by (3). There is obviously much freedom in this system, there being three equations for the four functions, but this may be removed by fixing the gauge freedom (see [7]). This is entirely analogous to the more familiar self-dual Yang–Mills equations $F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$, which are also a set of three equations for the four gauge potentials. Once again this freedom may be removed by fixing the gauge.

The Boyer–Finley equation (1) was first obtained by studying the reductions of the self-duality equations under a non-self-dual Killing vector (if the Killing vector is self-dual then the equations reduce to the three-dimensional Laplace equation) [2]. This result was

obtained by performing a dimensional reduction on Plebanski's equation (4) followed by a Legendre transformation.

One possible Lax pair for this equations, which clearly shows that this equation is a reduction of the self-dual Einstein equations, was proposed by Ward [10]. One may use the same vector fields

$$\mathcal{U} = (\partial_z + X_f) + \lambda X_{h_+}$$

$$\mathcal{V} = X_{h_-} - \lambda(\partial_{\tilde{z}} + X_g)$$

as before, but with the functions satisfying the following ansatz:

$$f = f(z, \tilde{z}, x)$$

$$g = g(z, \tilde{z}, x)$$

$$h_\pm = h(z, \tilde{z}, x) \exp(\pm p)$$

i.e. the functions f and g are independent of the variable p and the functions $h_\pm$ have a specific dependence on this variable. The condition that the two vector fields $\mathcal{U}$ and $\mathcal{V}$ commute implies the following set of equations (note that with this ansatz the variable p has dropped out leaving equations in the remaining variables z, $\tilde{z}$ and x):

$$\partial_z h - h \partial_x f = 0 \tag{8}$$

$$\partial_{\tilde{z}} h + h \partial_x g = 0 \tag{9}$$

$$\partial_z g - \partial_{\tilde{z}} f + 2h \partial_x h = 0. \tag{10}$$

These reduce to the Boyer–Finley equation (1), where $\rho = 2 \log h$. Owing to the use of the algebra sdiff (Σ^2) this equation has also been called the sdiff (Σ^2) Toda equation or the $SU(\infty)$ Toda equation. This establishes the bottom line in figure 1.

3. Deformations of sdiff (Σ^2)

Central to the above derivation is the identity $[X_f, X_g] = X_{\{f,g\}}$. The deformation that will be studied here is to replace the Poisson bracket by the Moyal bracket [11]

$$\{f, g\}_M = \sum_{s=0}^{\infty} \frac{\hbar^{2s}}{2^{2s}(2s+1)!} \sum_{j=0}^{2s+1} (-1)^j \binom{2s+1}{j} (\partial_x^{2s+1-j} \partial_p^j f)(\partial_x^j \partial_p^{2s+1-j} g). \tag{11}$$

This has the important properties

$$\{f, g\}_M = \frac{f \star g - g \star f}{\hbar}$$

$$\lim_{\hbar \to 0} \{f, g\}_M = \{f, g\}$$

$$\{\{f, g\}_M, h\}_M + \{\{g, h\}_M, f\}_M + \{\{h, f\}_M, g\}_M = 0$$

where $\star$ is an associative, but non-commutative, product which reduces to normal multiplication in the limit $\hbar \to 0$ defined by

$$f \star g = \sum_{s=0}^{\infty} \frac{\hbar^s}{2^s s!} \sum_{j=0}^{s} (-1)^j \binom{s}{j} (\partial_x^{s-j} \partial_p^j f)(\partial_x^j \partial_p^{s-j} g).$$

Such deformations are essentially unique [12]; any other deformation is ultimately equivalent to the Moyal bracket (11). One such equivalent deformation is given by

$$\{f, g\}_K = \sum_{s=1}^{\infty} \frac{\hbar^{s-1}}{s!} (\partial_x^s f \partial_p^s g - \partial_p^s f \partial_x^s g) \tag{12}$$

this being the algebra of symbols of pseudo-differential operators (see, for example, [13] and reference therein) and is known as the Kuperschmidt–Manin bracket. This may also be written in terms of an associative product, namely

$$f \star g = \sum_{s=0}^{\infty} \frac{\hbar^s}{s!} \partial_x^s f \, \partial_p^s g \, .$$

The dispersive self-dual Einstein equation is defined by using such brackets in place of the Poisson bracket in equation (4) and/or (5)–(7). Such a procedure may be made more geometrical in terms of a deformed differential geometry, but such an approach would take one beyond the scope of this paper [13]. As before, there are many different forms that this dispersive self-dual Einstein equation can take. One is [3]

$$\{\Omega_z, \Omega_{\tilde{z}}\}_M = 1 \tag{13}$$

this being a dispersive version of Plebanski's first heavenly equation. Further properties of this equation have been found [4, 5]. In particular, it has been shown that solutions may be encoded via a Riemann–Hilbert problem in the corresponding Moyal loop group. These results show the integrability of such Moyal deformations.

The version used in this paper follows from equations (5)–(7) with the Poisson bracket replaced by the Moyal bracket:

$$\partial_z h_- + \{f, h_-\}_M = 0 \tag{14}$$

$$\partial_{\tilde{z}} h_+ + \{g, h_+\}_M = 0 \tag{15}$$

$$\partial_z g - \partial_{\tilde{z}} f + \{f, g\}_M - \{h_+, h_-\}_M = 0 \, . \tag{16}$$

These equations follow from the Lax pair $[\mathcal{U}, \mathcal{V}] = 0$ with $[X_f, X_g] = X_{\{f,g\}_M}$. The methods used to show the integrability of equation (13) may be used to prove the integrability of this four dimensional system.

This is the system which appears in the top left-hand corner in figure 1. The property $\lim_{\hbar \to 0} \{f, g\}_M = \{f, g\}$ of the Moyal bracket establishes the left-hand side of figure 1 (as in this limit equations (14)–(16) reduce to equations (5)–(7)). It remains for us to show that this system contains, via a dimensional reduction, the Toda lattice equation.

4. Reduction to the Toda lattice equation

In section 2 it was shown that with the ansatz

$$f = f(z, \tilde{z}, x)$$

$$g = g(z, \tilde{z}, x)$$

$$h_\pm = h(z, \tilde{z}, x) \exp(\pm p)$$

the four-dimensional self-dual Einstein equations (5)–(7) reduce to the three dimensional Boyer–Finley equation. In this section the same ansatz will be used in the Moyal deformed version of these equations, the dispersive Einstein equations (14)–(16), and, as before, the variable p will drop out leaving equations in the remaining three variables. At first sight

this might seem a rather strange thing to do, as this will introduce an infinite number of extra terms into the governing equations, corresponding to the infinite summation in the definition of the Moyal bracket. However, it will turn out that it is possible to sum these infinite series and hence obtain results in closed form. The analogues of equations (8)–(10) are

$$\partial_z h - h \sum_{s=0}^{\infty} \frac{\hbar^{2s}}{2^{2s}(2s+1)!} \partial_x^{2s+1} f = 0$$

$$\partial_{\tilde{z}} h + h \sum_{s=0}^{\infty} \frac{\hbar^{2s}}{2^{2s}(2s+1)!} \partial_x^{2s+1} g = 0$$

$$\partial_z g - \partial_{\tilde{z}} f + \sum_{s=0}^{\infty} \frac{\hbar^{2s}}{2^{2s}(2s+1)!} \sum_{j=0}^{2s+1} \binom{2s+1}{j} \partial_x^{2s+1-j} h \, \partial_x^j h = 0$$

and these may be resummed in terms of a difference operator

$$\partial_z h(x) - h(x) \frac{f(x+\hbar/2) - f(x-\hbar/2)}{\hbar} = 0$$

$$\partial_{\tilde{z}} h(x) + h(x) \frac{g(x+\hbar/2) - g(x-\hbar/2)}{\hbar} = 0$$

$$\partial_z g(x) - \partial_{\tilde{z}} f(x) + \frac{h^2(x+\hbar/2) - h^2(x-\hbar/2)}{\hbar} = 0$$

(the dependence of these functions on the z, $\tilde{z}$ coordinates has been suspended for notational convenience). With $\rho = 2 \log h$ one obtains the Toda lattice equation

$$\nabla^2 \rho(x) = \hbar^{-2} \left[e^{\rho(x+\hbar)} - 2e^{\rho(x)} + e^{\rho(x-\hbar)} \right]. \tag{17}$$

So the same ansatz used to show that the Boyer–Finley equation is a reduction of the self-dual Einstein equations may also be used to show that the Toda lattice is a reduction of the dispersive self-dual Einstein equations. This establishes the top line in figure 1. On taking the limit $\hbar \to 0$ one recovers the Boyer–Finley equation, which is the remaining right-hand side of figure 1.

As remarked earlier, the Boyer–Finley equation was obtained directly from a dimensionally reduced version of Plebanski's equation (4) via a Legendre transformation. One might hope that there might be a direct transformation from a dimensionally reduced version of (13) to the Toda lattice (17), via some sort of Legendre transformation [2, 5].

These equations differ in one important respect to the normal form of the Toda-lattice equation (2); the variable x is continuous, not discrete. As was pointed out by Kupershmidt [14], taking the lattice to be $\mathbb{Z}$ is not essential, and the main properties will of the Toda lattice equation will hold in more general situations. For fixed $\hbar$ (say $\hbar = 1$) one may embed any solution if the Toda lattice equation (2) in the above equation (17), for example, for $x \in \mathbb{Z}$ let $\rho(z, \tilde{z}, x) = \rho_x(z, \tilde{z})$ and $\rho = 0$ elsewhere. However, whether such solutions can be 'smeared' over the whole of the domain of x, and how such solutions behave in the $\hbar \to 0$ limit requires careful analysis. Also, in general one cannot obtain a specific solution to the Boyer–Finley equation from a specific solution of the Toda lattice by scaling and taking the $\hbar \to 0$ limit. However, one particularly simple class of solutions for which this procedure does work is given by the ansatz

$$\rho(z, \tilde{z}, x) = \tilde{\rho}(z, \tilde{z}) + 2 \log x$$

where $\tilde{\rho}$ satisfies Liouville's equation. This satifies both the Boyer–Finley and the Toda lattice equations.

Using the alternative deformation of the Poisson bracket (12) one still obtains the Toda lattice equation, but by a slightly different route. The analogues of equations (8)–(10) are

$$\partial_z h(x) - h(x)\frac{f(x) - f(x - \hbar)}{\hbar} = 0$$

$$\partial_{\bar{z}} h(x) + h(x)\frac{g(x + \hbar) - g(x)}{\hbar} = 0$$

$$\partial_z g(x) - \partial_{\bar{z}} f(x) + h(x)\frac{h(x + \hbar) - h(x - \hbar)}{\hbar} = 0$$

with which one obtains (17), with $\rho = \log h(x) + \log h(x + \hbar)$. Thus with the Moyal bracket one has a central difference operator and with the Kupershmidt-Manin bracket (12) one has a forward/difference operator. Note that in both cases the deformation parameter $\hbar$ plays the role of the lattice spacing. This equivalence (between the forward/backward difference scheme and the central difference scheme) is well know and just corresponds to a gauge transformation. Here we see that this equivalence may also to traced back to the uniqueness theorems for associative $\star$-products [12].

5. Comments

The Toda-lattice equation (2) can be interpreted as the large N-limit of the $SU(N)$ Toda equation

$$\nabla^2 \rho_a = \sum_b K_{ab} e^{\rho_b} \tag{18}$$

where K_{ab} is the Cartan matrix of the $SU(N)$. For finite N, these equations may be embedded within the self-dual Yang–Mills equations with gauge group $SU(N)$. However, taking the large N-limit is problematical, especially when one tries to understand the integrability of these equations. For example, the Riemann–Hilbert problems used to construct solutions to the Toda lattice equation is very different in character to that used for the above, finite, Toda equation (for a review, see [15]). Alternatively, but presumably equivalently, the geometric constructions behind the above Toda equation and the Boyer–Finley equations are also somewhat different. The results of this paper suggests one should study an interim equation, the difference Toda equation (17), to understand the relationships between these equations; the limit that takes one from (18) to (1) is a double limit, first $N \to \infty$ then $\hbar \to 0$. Similar results apply to understanding the relationship between the KP equation and the dispersionless KP equation [16, 17].

These results show the connection between the Moyal algebra and the algebra of difference operators normally used in the study of the infinite Toda lattice. This relationship has been noticed recently, in various different contexts, by a number of authors and is the starting point for the construction of a deformed differential calculus [13, 18].

Finally, it must be noted that while all forms of the dispersionless self-dual Einstein equations are equivalent (i.e. one may go from any self-dual metric to any other via a coordinate transformation), it is by no means certain that their dispersive counterparts are so equivalent (e.g., while equations (4) and (5)–(7) are equivalent, it is not obvious that equations (13) and (14)–(16) are equivalent), and so the definite article in the title of this paper is, perhaps, premature. Thus what has been proved in this paper is the one particular form of the dispersive self-dual Einstein equations (one from from within a large class of

possible forms) contains the Toda lattice as a dimensional reduction. The equivalence, or otherwise, of these dispersive self-dual Einstein equations clearly deserves closer attention.

Acknowledgments

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